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BROWN-PETERSON AND ORDINARY COHOMOLOGY THEORIES OF CLASSIFYING SPACES FOR COMPACT LIE GROUPS

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Dedicated to Professor Tokushi Nakamura on his 60th birthday

ABSTRACT. The Steenrod algebra structures of $H^*(BG; Z/p)$ for compact Lie groups are studied. Using these, Brown-Peterson cohomology and Morava K-theory are computed for many concrete cases. All these cases have properties similar as torsion free Lie groups or finite groups, e.g., $BP^{\mathrm{odd}}(BG) = 0$.

INTRODUCTION

Let BG be the classifying space of a compact Lie group G. Let p be a fixed prime. It is well known that if $H^*(BG)_{(p)}$ has no p-torsion, then it is a polynomial algebra generated by even dimensional elements. Therefore the Atiyah-Hirzebruch type spectral sequence converging to the Brown-Peterson cohomology $BP^*(BG)$ collapses and $BP^*(BG) \simeq BP^* \otimes H^*(BG)_{(p)}$ where $\otimes$ denotes completed tensor product (see §1). Hence we get:

(1) $BP^*(BG) = BP^{\mathrm{even}}(BG)$.
(2) $BP^*(BG)$ is p-torsion free.
(3) $BP^*(BG)$ has no nilpotent elements.
(4) $BP^*(BG)$ is BP^*-flat for finite $BP^*(BP)$-modules. Moreover

$$BP^*(BG \times BG') \simeq BP^*(BG) \otimes_{BP^*} BP^*(BG')$$

for all compact Lie groups G'.
(5) $K(n)^*(BG) \simeq K(n)^* \otimes_{BP^*} BP^*(BG)$

where $K(n)^*(-)$ is the Morava K-theory. Moreover if G is a classical Lie group, we know

(6) $BP^*(BG) = \mathrm{Ch}_{BP}(BG)$, the Chern subring of $BP^*(BG)$ generated by Chern classes for all complex representations.

The main purpose of this paper is to show that the above properties hold in many cases even if $H^*(BG)$ has p-torsion. Note that for the ordinary cohomology theory $H^*(BG)_{(p)}$, the corresponding properties (1)–(4), (6) do not always hold, for example, $H^*(BG)_{(p)} \neq H^{\mathrm{even}}(BG)_{(p)}$. Landweber showed (1)–(6) hold for all abelian groups [L1]. Moreover he conjectured (2), (4), (6) for

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all compact Lie groups in [L3]. By [T-Y, Y2] when G is a direct product of metacyclic groups or minimal nonabelian p-groups (1)–(6) hold. A result of Hopkins, Kuhn and Ravenel [H-K-R] easily shows that when G is a finite group, (2) implies

$$BP^*(BG) \hookrightarrow \varprojlim BP^*(BA), \tag{7}$$
A runs through all conjugacy classes of abelian subgroups of G.

Remark that if a p-Sylow subgroup of a finite group G satisfies (1)–(7), so does G.

On the other hand, Wilson showed that $BP^*(BO(n))$ is generated by the Chern classes of the complexification of the universal real bundle. By using Wilson's arguments, we show

Theorem 1. *Properties* (1)–(2) *and properties* (4)–(6) *hold for direct products of* $O(n)$, $SO(2n+1)$.

The ordinary cohomology rings $H^*(BG; Z/p)$ for $G = F_4$, $PU(3)$ are given by Toda [T1] and by Kono, Mimura, and Shimada [K-M-S]. We study $H^*(BPU(3))$ in detail, considering the relation to its abelian subgroups. Hence we get

$$H^*(BG; Z/p) \hookrightarrow \varprojlim H^*(BA; Z/p) \tag{7$'$}$$

for $G = PU(3)$. This was conjectured by J. F. Adams for all connected compact Lie groups G and $p \geq 3$ and solved for $G = F_4$, $p = 3$ by Adams and Kono. We also know that there are only two conjugacy classes of maximal elementary 3-abelian subgroups of $PU(3)$. Moreover we can determine the Steenrod algebra structure of $H^*(BPU(3))$. Using these, we show

Theorem 2. *Properties* (1)–(5), (7) *hold when* $G = PU(3)$ *and* F_4 *for* $p = 3$, *but* (6) *does not hold for* $G = PU(3)$.

Mimura and Kono study $H^*(BG; Z/p)$ for many compact Lie groups [K-M 1, 2]. Also using their results, we get

Theorem 3. *The properties* (1)–(3), (7) *hold when* $G = \mathrm{Spin}(n) n \leq 10$, G_2, F_4, E_6, $PSU(4n+2)$ *for* $p = 2$.

Bakuradze [B-N] showed that (1)–(7) hold for the normalizer group of maximal torus in $Sp(1) \times Sp(1)$. Hunton showed $K(n)^*(BG) = K(n)^{\text{even}}(BG)$ for some other compact Lie groups [H]. Inoue [I] determined $BP^*(BSO(6))$ and showed (1) for this case.

Conjecture 4. Assertions (1)–(5) and (7) hold for all compact Lie groups.

There are no application of these results now. However we hope $BP^*(BG)$ can aid in the understanding of the ordinary cohomology $H^*(BG)$ which seems so complicated in general cases. For example, we presume that the following conjecture, which holds in all cases in Theorems 1–3, is true.

Conjecture 5. If G is a connected compact Lie group, then for each odd dimensional element $x \in H^*(BG; Z/p)$, there is i such that $Q_m x \neq 0$ for all $m \geq i$, where Q_m are the Milnor primitive operators.

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1. BP AND RELATED COHOMOLOGY THEORIES

Throughout this paper, we assume that spaces X, Y mean CW-complexes whose n-skeleton is finite complexes for each $n \geq 0$. Let $BP^*(-)$ be the Brown-Peterson cohomology localized at p with the coefficient $BP^* = Z_{(p)}[v_1, \dots]$, $|v_i| = -2(p^i - 1)$. We consider the cohomology theories $k^*(-)$ with the coefficient $k^* = BP^*/(\text{Ideal } S)$, e.g., $P(n)^* = Z/p[v_n, v_{n+1}, \dots]$, $\tilde{P}(n)^* = Z_{(p)}[v_n, \dots]$, $BP\langle n\rangle^* = Z_{(p)}[v_1, \dots, v_n]$, $k(n)^* = Z/p[v_n]$, $\tilde{k}(n)^* = Z_{(p)}[v_n]$, and $K(n)^* = Z/p[v_n, v_n^{-1}]$. We consider the Atiyah-Hirzebruch spectral sequence $E_2^{*,*} = H^*(X; k^*) \Rightarrow k^*(X)$. Hereafter we assume the convergence of this spectral sequence and hence

(1.1) $$j^*: k^*(X) \simeq \varprojlim_N k^*(X^N).$$

Note that if $X = BG$ or k^n is a fine group, this assumption holds. (See [L3].)

Define a filtration $F^N(X) = \operatorname{Ker}(j_N^*: k^*(X) \to k^*(X^N))$ of $k^*(X)$ and define a topology in $k^*(X)$ by $F^N(X)$ as the fundamental neighbourhoods of 0. Then

(1.2) $$k^*(X) \simeq \varprojlim_N k^*(X)/F^N(X)$$

is a complete algebra. Let A, B be k^*-complete algebras with filtrations A^N, B^N. We define the complete tensor product $\otimes$ by

(1.3) $$A \otimes_{k^*} B \simeq \varprojlim A \tilde{\otimes}_{k^*} B/(A^N \tilde{\otimes}_{k^*} B + A \tilde{\otimes}_{k^*} B^N)$$

where $\tilde{\otimes}$ is the usual tensor product. Then if X and Y are p-torsion free, then we can write

(1.4) $$k^*(X \times Y) \simeq k^*(X) \otimes_{k^*} k^*(Y).$$

Note that $- \otimes_{k^*} -$ in this paper means that each element is expressed as infinite sum.

Landweber's exact functor theorem [L2] says that injectivity of the following (1.5) for all $n \geq 0$ (let $p = v_0$),

(1.5) $$v_n: P(n)^* \otimes_{BP^*} BP^*(X) \to P(n)^* \otimes_{P(n)^*} BP^*(X)$$

implies the BP^*-flatness of $BP^*(X)$ for finite $BP^*(BP)$-modules. In particular we have

(1.6) $$BP^*(X \times Y) \simeq BP^*(X) \otimes_{BP^*} BP^*(Y) \text{ for all } Y \text{ satisfies (1.1).}$$

From the Sullivan exact triangles

$$\begin{array}{ccccccc} BP^*(X) & \xrightarrow{\rho} & P(1)^*(X) & \xrightarrow{\rho} & P(2)^*(X) & \xrightarrow{\rho} & P(3)^*(X)\cdots \\ {\scriptstyle p}\nwarrow & \swarrow{\scriptstyle \delta} & {\scriptstyle v_1}\nwarrow & \swarrow{\scriptstyle \delta} & {\scriptstyle v_2}\nwarrow & \swarrow{\scriptstyle \delta} & \\ BP^*(X) & & P(1)^*(X) & & P(2)^*(X) & & \end{array}$$

the injectivity of (1.5) is equivalent to the assertion that $\rho: BP^*(X) \to P(n)^*(X)$ is epic for all $n \geq 0$ and is that

(1.7) $$P(n)^*(X) \simeq P(n)^* \otimes_{BP^*} BP^*(X).$$

From Johnson-Wilson theorem [J-W], if X satisfies (1.7), then we get

$$K(n)^*(X) \simeq K(n)^* \otimes_{BP^*} BP^*(X). \tag{1.8}$$

Lemma 1.9. *If X and Y satisfy the injectivity of (1.5), then so does $X \times Y$.*

Proof. By the exact functor theorem for $P(n)^*$-theory, we have

$$P(n)^*(X \times Y) \simeq P(n)^*(X) \otimes_{P(n)^*} P(n)^*(Y),$$

which is isomorphic to

$$P(n)^* \otimes_{BP^*} (BP^*(X)) \otimes_{P(n)^*} P(n)^* \otimes_{BP^*} (BP^*(Y)) \simeq P(n)^* \otimes_{BP^*} BP^*(X \times Y).$$

Therefore $X \times Y$ satisfies (1.7) and so (1.5). □

By the same argument as Theorem 3.3 [Y2], the kernel of $r\colon BP^*(BG) \to \varprojlim BP^*(BA)$; A all abelian subgroups, is nilpotent.

Lemma 1.10. *The injectivity of r is equivalent to that $BP^*(BG)$ has no nonzero nilpotent element.*

Proof. We only need $BP^*(BA)$ has no nilpotent element. Consider the spectral sequence induced from

$$0 \to A' \to A \to Z/p \to 0.$$

Then

$$E_2^{*,*'} \simeq E_\infty^{*,*'} = H^*(Z/p) \otimes BP^{*'}(A')/(p) \simeq Z/p[y] \otimes BP^{*'}(A')$$

for $* > 0$, since $BP^*(A')$ is p-torsion free. Suppose $a^l = 0$ in $BP^*(BA)$.

Let $y^s x \neq 0 \in E_\infty^{2s,*}$ be the corresponding element to a. Since $(y^s x)^l = 0$, $x^l = 0$ in $BP^*(BA')/(p)$. Let us write $x^l = p^r x'$, $x' \neq 0 \bmod p$ in $BP^*(BA')$. Then

$$a^l = p^r x' y^s = v_1^r y^{s+r(p-1)} x' \quad \text{in } E_\infty^{2(s+r(p-1)),*}.$$

This element is nonzero because $BP^*(BA')/(p)$ has no v_1-torsion. □

Therefore we get the following implications:

$$\begin{array}{c} (3) \Leftrightarrow (7) \Rightarrow (2) \Leftarrow ((1.7) \text{ for } X = BG) \Leftrightarrow (4) \Rightarrow (5) \\ \searrow (1) \Leftarrow (6). \end{array} \tag{1.11}$$

Kuhn, Hopkins, and Ravenel showed that when G is a finite group, $|G|^{-1}r$ is isomorphic. Therefore

$$(2) \Leftrightarrow (7) \quad \text{for finite groups.} \tag{1.12}$$

2. The orthogonal group $O(n)$

Before considering $BP^*(BO(n))$, we consider cohomology operations Q_i. Recall $Q_{i+1} = \mathscr{P}^{p^i} Q_i - Q_i \mathscr{P}^{p^i}$ $(= \mathrm{Sq}^{2^i} Q_1 + Q_i \mathrm{Sq}^{2^i}$ for $p = 2)$ and $Q_0 = \mathscr{B}$ $(Q_0 = \mathrm{Sq}^1)$. The first nonzero differential of the spectral sequence

$$E_2^{*,*} = H^*(X; P(m)^*) \Rightarrow P(m)^*(X) \tag{2.1}$$

is given by $d_{2p^m-1} = v_m \otimes Q_m$.

Lemma 2.2. *Let $E_i = \Lambda[Q_m, \ldots, Q_{m+i-1}]$ and $E_0 = Z/p$. Suppose that there is an E_i-module injective $E_i \otimes G_i \subset H^*(X; Z/p)$ and there is a Z/p-module isomorphism*

$$H^*(X; Z/p) \simeq \bigoplus_{i=0}^{M} E_i \otimes G_i$$

such that $Q_m \cdots Q_{m+i-1} G_i \in \operatorname{Im} \rho(P(m)^*(X) \to H^*(X; Z/p))$ (for $i = 0$ $G_0 \in$ $\operatorname{Im} \rho$). *Then the infinite term of the spectral sequence* (2.1) *is*

$$E_\infty^{*,*} \simeq \bigoplus_{i>0}^{M} P(i+m)^* Q_m \cdots Q_{m+i-1} G_i \oplus P(m)^* G_0.$$

Proof. By the induction on r for $d_{2p^{m+r-1}-1}$, we assume that $E_{2p^{m+r-1}} = A_r \oplus B_r$, where

$$A_r = \bigoplus_{i=0}^{r} P(m+i)^* Q_m \cdots Q_{m+i-1} G_i,$$

$$B_r = \bigoplus_{i=r+1}^{M} P(m+r)^* Q_m \cdots Q_{m+r-1}(E_{m+r,i-r}) G_i,$$

and where $E_{m+r,i-r} = \Lambda[Q_{m+r}, \ldots, Q_{m+i-1}]$. Indeed, $A_0 = P(n)^* G_0$ and $B_0 = P(m)^* \otimes (\bigoplus_{i=1}^{M} E_i \otimes G_i)$, hence $A_0 \oplus B_0 = P(m)^* \otimes H^*(X; Z/p)$.

By the supposition of the lemma, all elements in A are infinite cycles. Assume $d_s x \neq 0$, $x \in B$. Since B is a $P(m+r)^*$-free modules, it is necessary $s \geq 2p^{m+r} - 1$. Hence consider $d_{2p^{m+r}-1} = v_{m+r} \otimes Q_{m+r}$. Therefore

$$\begin{aligned} E_{2p^{m+r}} &\simeq A_r \oplus \bigoplus_{i=r+1}^{M} P(m+r+1)^* Q_m \cdots Q_{m+r}(E_{m+r+1,i-r-1}) G_i \\ &\simeq A_{r+1} \oplus B_{r+1}. \end{aligned}$$

Since $B_{M+1} = 0$, we get the lemma. □

If $H^*(X; Z/p) = E_{0,n} = \Lambda(Q_0, \ldots, Q_n)$, then

$$P(m)^*(X) \simeq P(n)^* \Lambda[Q_0, \ldots, Q_{m-1}].$$

This fact is known as $X = V(n)$, Smith-Toda spectrum.

The BP-cohomology of the classifying space of the nth orthogonal group, $BP^*(BO(n))$ is computed by W. S. Wilson. Since $H^*(BO(n))$ has only 2-torsion, we need only consider the 2-primary part.

Theorem 2.3 (Wilson [W]).

$$BP^*(BO(n)) \simeq BP^*[[c_1, \ldots, c_n]]/(c_1 - c_1^*, \ldots, c_n - c_n^*)$$

where c_i is the ith Chern class of complexification of universal real bundle and c_i^* is the Chern class of its complex conjugation.

Recall the $Z/2$-cohomology of $BO(n)$ and $(BZ/2)^n$. It is well known

$$\begin{array}{ccc} H^*(BO(n); Z/2) & \hookrightarrow & H^*((BZ/2)^n; Z/2) \\ \wr\| & & \wr\| \\ Z/2[w_1, \ldots, w_2] & \hookrightarrow & Z/2[x_1, \ldots, x_n] \end{array}$$

where w_i is the ith elementary symmetric polynomial of x_s. Then w_i is the ith Whitney class and $c_i = i^*(c_i) = w_i^2$ for the complexification map $i\colon BO(n) \to BU(n)$. The following lemma is just the $P(m)^*$-analogue of Wilson's (p. 359, Theorem 2.1 in [W]).

Lemma 2.4. *Let G_k be $Z/2$-vector space in $H^*(BO(n); Z/2)$ generated by symmetric functions*

$$\sum x_1^{2i_1+1} \cdots x_k^{2i_k+1} x_{k+1}^{2j_1} \cdots x_{k+q}^{2j_q}, \qquad k+q \le n,$$

with $0 \le i_1 \le \cdots \le i_k$ and $0 \le j_1 \le \cdots \le j_q$; and if the number of j equal to j_u is odd, then there is some $s \le k$ such that

$$2i_s + 2^{s+m} < 2j_u < 2i_s + 2^{s+m+1}.$$

Then G_k satisfies the assumption of Lemma 2.2 and hence the infinite term of the Atiyah-Hirzebruch spectral sequence converging to $P(m)^*(BO(n))$ is

$$E_\infty^{*,*} \simeq \bigoplus_{i=0}^{n} P(m+r)^* Q_m \cdots Q_{m+r} G_r.$$

Proof. First note $Q_m \cdots Q_{m+r} G_r$ is generated by functions of $\sum x_1^{2h_1} \cdots x_{k+q}^{2h_{k+q}}$ and hence is in $Z/2[w_1^2, \ldots, w_n^2]$ which is in

$$\operatorname{Im} \rho(P(m)^* BO(n)) \to H^*(BO(n); Z/2))$$

since $i^*(c_j) = w_j^2$. The proof for satisfying the assumption of Lemma 2.2 is completely the same as the proof of Wilson's Theorem 1 [W, p. 360] except for changing Q_i to Q_{m+i} and 2^{v+1} to 2^{v+m}. □

Corollary 2.5.

$$P(m)^*(BO(n)) \simeq P(m)^* \otimes_{BP^*} BP^*(BO(n)),$$
$$K(m)^*(BO(n)) \simeq K(m)^* \otimes_{BP^*} BP^*(BO(n)).$$

Proof. From Lemma 2.4, $\rho\colon BP^*(BO(n)) \to P(m)^*(BO(n))$ is epic. □

It is well known that there is an isomorphism of Lie groups

$$Z/2 \times SO(2n+1) \simeq O(2n+1)$$

and hence

$$BP^*(BZ/2) \otimes_{BP^*} BP^*(BSO(2n+1)) \simeq BP^*(BO(2n+1)),$$

and $BP^*(BZ/2)$ is BP^*-flat. Therefore $BP(BSO(2n+1))$ is generated by c_i $(= w_i^2)$, $2 \le i \le 2n+1$. The same facts hold for $P(m)^*$-theory $n \ge 1$. Hence $\rho\colon BP^*(BSO(2n+1)) \to P(m)^*(BSO(2n+1))$ is epic. Therefore we get Theorem 1 in the introduction.

Remark that the squaring operation is given by

$$\text{(2.6)} \qquad \mathrm{Sq}^i w_k = \sum_j^i \binom{k-j-1}{i-j} w_{k+i-j} w_j \qquad (0 \le i \le k).$$

3. Cohomology of $BPU(3)$

The projective unitary group $PU(3)$ is defined as $PU(3) = SU(3)/\Gamma$ where Γ is the center of $SU(3)$. Let us write

$$(3.1)\qquad \tilde{a} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \tilde{b} = \begin{pmatrix} w & 0 & 0 \\ 0 & w^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \tilde{c} = \begin{pmatrix} w & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & w \end{pmatrix}$$

where $w = \exp(2\pi i/3)$. Note that $\Gamma = \langle \tilde{c} \rangle$. The group generated by $\langle \tilde{a}, \tilde{b}, \tilde{c} \rangle$ is E, the nonabelian 3-group of order 27 with its exponent 3. Consider the elementary abelian 3-subgroups $V_1 = E/\Gamma = \langle \tilde{a} \rangle \oplus \langle \tilde{b} \rangle$ and $V_2 \subset T^2$, the maximal torus of $PU(3)$.

Quillen [Q1] proved that for a compact Lie group

$$(3.2)\qquad r\colon H^*(BG; Z/p) \simeq \varprojlim_V H^*(BV; Z/p)$$

is an F-isomorphism, where V runs the conjugacy classes of elementary p-subgroups of G. Here an F-isomorphism means $\operatorname{Ker} r \subset \sqrt{0}$; nilpotent elements and for each x, $x^{p^s} \in \operatorname{Im} r$ for some s.

We will prove a much stronger result for $G = PU(3)$.

Theorem 3.3. *The restriction map*

$$r\colon H^*(BPU(3); Z/3) \to H^*(BV_1; Z/3) \otimes H^*(BV_2; Z/3)$$

is injective.

Therefore $\{V_1, V_2\}$ is the set of the conjugacy classes of maximal elementary 3-subgroups in $PU(3)$.

Let ρ be the canonical representation in $SU(3)$ and

$$(3.4)\qquad \tilde{\lambda} = \rho \otimes \rho^{-1}\colon SU(3) \to SU(9).$$

Since $\rho \otimes \rho^{-1}|\Gamma$ is trivial, $\tilde{\lambda}$ induces the representation λ of $PU(3)$. It is easily computed.

Lemma 3.5.

$$\chi_{\tilde{\lambda}}(a^i b^j) = \begin{cases} 9, & i \equiv 0,\ j \equiv 0 \bmod 3, \\ 0, & \text{otherwise.} \end{cases}$$

Corollary 3.6. *$\lambda|V_1$ is the regular representation.*

Think of $PU(3)$ as $U(3)/\widetilde{\Gamma}$ where $\widetilde{\Gamma} = \{\text{diagonal matrix } (\alpha, \alpha, \alpha),\ \alpha \in S^1\}$ and $\pi\colon U(3) \to PU(3)$ is its projection map. Let $\widetilde{T}$ and $\pi(\widetilde{T}) = T$ be the maximal tori in $U(3)$ and $PU(3)$.

The fundamental class $\pi_1(\widetilde{T})$ is generated as $\langle \tilde{t}_1, \tilde{t}_2, \tilde{t}_3 \rangle$ where

$$\tilde{t}_1 = \{\text{diagonal}(\exp 2\pi i t, 1, 1),\ t \in [0, 1]\}$$

and so on. Then it is easily seen $\pi_*(\tilde{t}_1 + \tilde{t}_2 + \tilde{t}_3) = 0$ and $\operatorname{Ker} \pi_* = \langle \tilde{t}_1 + \tilde{t}_2 + \tilde{t}_3 \rangle$. Denote by $t_j \in H^2(B\widetilde{T}; Z) \simeq H^1(\widetilde{T}) \simeq H_1(\widetilde{T})$ corresponding to $\tilde{t}_j$. Let us write $u, v \in H^2(BT; Z)$ the corresponding elements to $\pi_*(\tilde{t}_1)$ and $\pi_*(\tilde{t}_2)$ respectively.

Lemma 3.7. *$\pi^*u = t_1 - t_3$ and $\pi^*v = t_2 - t_3$.*

Proof. Since $H^*(BT; Z) \to H^*(B\widetilde{T}; Z)$ is monic we get the lemma from

$$\langle \pi^*u, \tilde{t}_i\rangle = \langle u, \pi_*\tilde{t}_i\rangle = 1 \quad (\text{resp. } 0, -1), \qquad i = 1 \quad (\text{resp. } = 2, = 3).$$

Here we note $\pi_*(\tilde{t}_3)$ corresponds to $-u - v$. □

Let us write T_9 be the maximal torus of $U(9)$ and $\pi_1(T_9) = \langle t_{ij} | 1 \le i, j \le 3\rangle$. Then $\tilde{\lambda}^*(t_{ij}) = t_i - t_j$. Since $\pi^*: H^*(BT; Z) \to H^*(B\widetilde{T}; Z)$ is monic, $\lambda^*(t_{ij}) = -\lambda^*(t_{ji})$ and $\lambda^*(t_{12}) = u - v$, $\lambda^*(t_{13}) = u$, $\lambda^*(t_{23}) = v$.

Lemma 3.8. *The total Chern class in $H^*(BT)$ for $\lambda|T$ is*

$$C(\lambda|T) = \lambda^*(\pi(1 + t_{ij})) = (1 - u^2)(1 - v^2)(1 - (u - v)^2)$$
$$= 1 + (u + v)^2 + (u + v)^4 - u^2v^2(u^2 + uv + v^2).$$

From Corollary 3.6,

Lemma 3.9.

$$C(\lambda|V_1) = \pi(1 + \lambda_1 a + \lambda_2 b)$$
$$= (1 - a^2)(1 - b^2)(1 - (a + b)^2)(1 - (a - b)^2)$$
$$= 1 - (a^6 + a^4b^2 + a^2b^4 + b^6) + a^2b^2(a^4 + a^2b^2 + b^4)$$

where $a, b \in H^2(BV_1; Z/3)$ is the Bockstein image of the dual element of $\tilde{a}$ and $\tilde{b}$, identifying $H^1(BV_1; Z/3) \simeq \mathrm{Hom}(V_1; Z/3)$.

The cohomology of $BPU(3)$ is given by Kono, Mimura, and Shimada [K-M-S].

Theorem 3.10. *There is an algebra isomorphism*

$$H^*(BPU(3); Z/3) \simeq Z/3[y_2, y_8, y_{12}] \otimes \Lambda(y_3, y_7)/J$$

where $|y_i| = i$ and J is the ideal generated by y_2y_3, y_2y_7 and $y_3y_7 + y_2y_8$. Moreover $y_3 = \mathscr{B}y_2$, $y_7 = \mathscr{P}y_3$, and $y_8 = \mathscr{B}y_7$.

Note that $y_2^2y_8 = -y_3y_2y_7 = 0$. Let $R_1 = Z/3[y_2, y_{12}]$ and $R_2 = Z/3[y_8, y_{12}]$. Then there are $Z/3$-modules isomorphism

$$H^*(BPU(3); Z/3) \simeq y_2^2R_1 \oplus Z/3\{1, y_2, y_3, y_7\}R_2. \tag{3.11}$$

Lemma 3.12. *The ideal generated by y_2^2 is $y_2^2R_1$ in* (3.11).

Consider the induced map $j_1: BV_1 \to BPU(3)$. Let a', b' be the dual element of $\tilde{a}, \tilde{b}$ in $H^1(V_1; Z/3) \simeq \mathrm{Hom}(V_1; Z/3)$ and $\mathscr{B}a' = a$, $\mathscr{B}b' = b$. The commutative diagram

$$\begin{array}{ccccc} B\Gamma & \longrightarrow & BSU(3) & \longrightarrow & BPU(3) \\ \| & & \uparrow & & \uparrow{\scriptstyle j_1} \\ B\Gamma & \longrightarrow & BE & \longrightarrow & BV_1 \end{array} \tag{3.13}$$

induces the map of spectral sequences

$$E_2^{*,*} = H^*(BV_1; H^*(B\Gamma; Z/3)) \xleftarrow{j_1^*} \widetilde{E}_2^{*,*} = H^*(BPU(3); H^*(B\Gamma; Z/3)).$$

Since $H^*(BSU(3))$ and $H^*(BE)$ is known, we get $d_2c' = a'b'$ in $E_2^{*,*}$ and $d_2c' = y_2$ in $\widetilde{E}_2^{*,*}$. Therefore $j_1^*y_2 = a'b'$.

Lemma 3.14. *$j_1^* y_3 = \beta(a'b') = ab' - a'b$, $j_1^* y_7 = a^3b' - a'b^3$, and $j_1^* y_8 = a^3b - ab^3$.*

The restriction $y_i|\langle a\rangle = 0$ for $i \neq 12$, but $c_6(\lambda)|\langle b\rangle = b^6 \neq 0$ from Lemma 3.9. Hence we can take $y_{12} = -c_6(\lambda)$.

Lemma 3.15. $\mathrm{Ker}\, j_1^* = y_2^2 R_1$.

Proof. We need only prove $j_1^*|(y_3R_1 + y_7R_3)$ is monic. From Lemmas 3.12 and 3.14, $(j_1^* y_8, j_1^* y_{12})$ is a regular sequence in $Z/3[a, b]$. Therefore $j_1^* f(y_8, y_{12}) = 0$ implies $f \equiv 0$. Let $j_1^*(y_3 f + y_7 g) = 0$ and $j_1^* f = F$, $j_1^* g = G$. Taking modulo a', we get $aF + a^3G = 0$ and taking modulo b', we have $bF + b^3G = 0$. This implies $ab(a^2 - b^2)G = 0$, hence $G = 0$. The regularity follows $g = 0$. By the same argument, we also get $f = 0$. Of course $j_1^* y_2^2 = (a'b')^2 = 0$. □

Let $j_2\colon BT \to BPU(3)$ be the map from the inclusion of the maximal torus. Since $H^*(BT; Z)$ is torsion free, and is generated by even dimensional elements, we have

(3.16) $$j_2^* y_3 = j_2^* y_7 = j_2^* y_8 = 0.$$

Since $j_2^* c_2(\lambda) = (u + v)^2 \neq 0$, we can take $j_2^* y_2 = u + v$. Therefore from Lemma 3.9,

(3.17) $$c_1(\lambda) = 0, \qquad c_2(\lambda) = \varepsilon y_2^2, \qquad \varepsilon = \pm 1, \qquad c_3(\lambda) = 0.$$

The formula $c_4 = \mathscr{P}^2 c_2 + c_3 c_1 - c_2(c_2 + c_1^2) = \mathscr{P}^2 c_2 - c_2^2$ implies $y_2^4 = (2\varepsilon - 1) y_2^4$. Hence we get $\varepsilon = 1$.

Lemma 3.18. *The Chern classes $c_j(\lambda)$ are represented by y_2^2, y_2^4, y_{12}, and y_8^2 for $j = 2, 4, 6$, and 8, respectively.*

Comparing Lemmas 3.8 and 3.9 and considering the diagram

$$\begin{array}{ccccccc} E & \xrightarrow{i_1} & U(3) & \xleftarrow{i_2} & \widetilde{T} & & \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_2} & & \\ V_1 & \xrightarrow[j_1]{} & PU(3) & \xleftarrow[j_2']{} & T & \xleftarrow[j_2'']{} & V_2 \end{array}$$

we have a short exact sequence

(3.19) $$0 \to \pi^*(\mathrm{Ker}\, j_1^*) \rightleftarrows H^*(BPU(3); Z/3) \to \mathrm{Ker}\, \pi_1^* \to 0$$

with $\pi^*(\mathrm{Ker}\, j_1^*) = Z/3[c_1, c_6]\{c_2\} \subset H^*(BU(3); Z/3)$ and $\mathrm{Ker}\, \pi_1^* = Z/3\{\mathscr{P}^I a'b'\} \subset H^*(V_1; Z/3)$.

Using Lemma 3.18, (3.19), and Lemma 3.14, we decide the Steenrod algebra structure of $H^*(BPU(3); Z/3)$, and have proved Theorem 3.3.

Theorem 3.20. *$\mathscr{P}^1 y_3 = y_7$, $\mathscr{B} y_7 = y_8$, $\mathscr{P}^3 y_7 = y_7 y_{12} + y_3 y_8^2$, $\mathscr{P}^3 y_8 = y_8 y_{12}$, $\mathscr{P}^1 y_{12} = y_8^2 + y_{12} y_2^2$, $\mathscr{P}^3 y_{12} = y_{12}(y_2^6 - y_{12})$.*

4. Brown-Peterson cohomology of $BPU(3)$

Recall (3.11) in §3,

$$A = H^*(BPU(3); Z/3) \simeq y_2^2 R_1 \oplus Z/3\{1, y_2, y_3, y_7\} R_2$$

where $R_1 = Z/3[y_2, y_{12}]$, $R_2 = Z/3[y_8, y_{12}]$ and $Q_0y_2 = y_3$, $Q_0y_7 = y_8$. Then $H(A; Q_0) \simeq y_2^2R_1 \oplus Z/2[y_{12}]$ and its Poincaré series is

$$\text{P.S.}(H(A; Q_0)) = \frac{t^4}{(1-t^2)(1-t^{12})} + \frac{1}{(1-t^{12})} = \frac{1}{(1-t^4)(1-t^6)},$$

which is the Poincaré series of the polynomial algebra of degree 4 and 6 and is equal to the Poincaré series of $H^*(BPU(3); Q_0)$. Therefore the Bockstein spectral sequence collapses, i.e., $E_1 \simeq E_\infty$. This means $H^*(BPU(3))$ has no higher 3-torsion.

Consider the Atiyah-Hirzebruch spectral sequence

(4.1) $$E_2^{*,*} = H^*(BPU(3); BP^*) \Rightarrow BP^*(BPU(3)).$$

The E_2-term is given by

(4.2) $$E_2^{*,*} = BP^* \otimes \{y_2^2\widetilde{R}_1 \oplus \widetilde{R}_3 \oplus R_2y_3 \oplus R_2y_8\}$$

where $\widetilde{R}_1 = Z_{(3)}[y_2, y_{12}]$, $\widetilde{R}_3 = Z_{(3)}[y_{12}]$. The first nonzero differential is $d_{2p-1} = v_1 \otimes Q_1$. Since $Q_1y_3 = y_8$, we get

(4.3) $$E_{2p}^{*,*} \simeq BP^*\{y_2^2\widetilde{R}_1 \oplus \widetilde{R}_3\} \oplus BP^*/(3, v_1) \otimes \{R_2y_8\}.$$

These are all even dimensional elements and $E_{2p}^{*,*} \simeq E_\infty^{*,*}$.

Therefore we see $BP^*(BPU(3)) = BP^{\text{even}}(BPU(3))$. Moreover each element in $E_{2p}^{*,*}$ is nonnilpotent, we get $BP^*(BPU(3))$ has no nonzero nilpotent element. Hence (1)–(3) and (7) in the introduction hold.

However (6) does not hold as following. Recall the representation ring $R(SU(3)) = Z[\iota, \bar\iota]$ where ι is represented by the character of the canonical representation ρ and $\bar\iota$ is its conjugation. The representation ring of $PSU(3)$ is easily deduced as the subring generated by

(4.4) $$\{\iota^c\bar\iota^d \mid c + 2d \equiv 0 \bmod 3,\ c, d \in \mathbb{Z}\}.$$

For $c, d \geq 0$, let $M(c, d)$ is the corresponding $PSU(3)$-module.

Lemma 4.5.

$$\chi_{M(c,d)}(\tilde a^i\tilde b^j) = \begin{cases} 3^{c+d}, & i \equiv j \equiv 0 \bmod 3, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. This is easily seen from the facts $T_r(f \times g) = T_r(f)T_r(g)$ and

$$\chi_i(\bar a^j) = \chi_i(\bar b^j) = \begin{cases} 3, & j \equiv 0 \bmod 3, \\ 0, & \text{otherwise.} \end{cases} \quad \square$$

Corollary 4.6. $M(c, d)|V_1 = 3^{c+d-2}$ (*regular representation*).

From Lemma 3.9, $y_8|V_1 \notin$ Image $\lambda^*(H^*(BU(3)) \to H^*(BV_1))$ where λ is the regular representation.

Corollary 4.7. *Since* $y_8 \notin \mathrm{Ch}_{BP}(BPU(3))$ *and* $y_2^3 \notin \mathrm{Ch}_{BP}(BPU(3))$, *we have*

$$\mathrm{Ch}_{BP}(BPU(3)) \neq BP^*(BPU(3)).$$

We now consider the BP^*-module structure of $BP^*(BPU(3))$. The BP^*-algebra structure of BZ/p is well known:

$$BP^*[[u]]/[p](u), \qquad |u| = 2,$$

where $[p](u) = u +_{BP} \cdots +_{BP} u$ is the pth product of the formal group law over BP^*-theory. Note that $[p](u) = \sum v_n u^{p^n} \mod (p, v_1, \dots)^2$ and $Q_n(\alpha) = \rho(u)^{p^n}$ where $H^*(BZ/p; Z/p) \simeq Z/p[\rho(u)] \otimes \Lambda(\alpha)$ and $\rho\colon BP \to Z/p$ be the natural map. This fact extends as the following lemma.

Lemma 4.8 [Y1]. *If there is a relation* $\sum v_n x_n = 0$ *in* $BP^*(X)$, *then there exists* $y \in H^*(X; Z/p)$ *such that* $Q_i(y) = \rho(x_i)$.

Theorem 4.9. *Let us fix elements* $\tilde{f}(y_i) \in BP^*(BPU(3))$ *with* $\rho(\tilde{f}) = f$, *i.e.,* $\rho(\tilde{y}_{12}) = y_{12}$, $\rho(\tilde{y}_2^2) = y_2^2$. *There is a* BP^**-module isomorphism* $BP^*(BPU(3)) \simeq (BP^*S_2\tilde{y}_2^2 \oplus BP^*\{1\} \oplus BP^*S_8\tilde{y}_8) \otimes S_{12}/(I_1, I_2)$ *where* $S_i = Z_{(3)}[[\tilde{y}_i]]$ *and*

$$I_1 \equiv \sum v_n \widetilde{Q}_n(y_7), \qquad I_2 \equiv \sum v_n \widetilde{Q}_n(y_3) \mod (3, v_1, \dots)^2.$$

Proof. From (4.3), there are relations such that $I_1 = p\tilde{y}_8 + \cdots$ and $I_2 = v_1\tilde{y}_8 + \cdots$. Since $\operatorname{Ker}\rho(BP^*(BPU(3)) \to H^*(BPU(3); Z/3)) = (v_1, v_2, \dots)$, we have the theorem from Lemma 4.8 and $Q_0 y_7 = y_8$, $Q_1 y_3 = y_8$. □

Lemma 4.10. *Let us write* $e_i = Q_i y_3$ *and* $X = y_8^3$, $Y = y_{12}^2$. *For* $i = 2j+1 > 0$, $e_i = f_i(X, Y)y_8$ *and* $e_{i+1} = g_{i+1}(X, Y)y_8 y_{12}$. *Moreover,*

$$g_{i+1} = (f_i')^3 Y X^2 + f_i^3, \qquad f_{i+2} = (g_{i+1}')^3 Y^3 X + g_{i+1}^3 (Y^2 + X^2),$$

where $f' = \partial f/\partial Y$ *and* $g' = \partial g/\partial Y$. *In particular,* $e_i = 0$, y_8, $y_{12}y_8$, $y_8(X^2 + Y^2)$ *for* $i = 0, 1, 2, 3$, *respectively.*

Proof. Let us denote $\mathscr{P}^{\#}a$ by $\# = 1/2|a|$. For $i \geq 1$,

$$e_{i+1} = Q_{i+1} y_3 = (\mathscr{P}^{p^i} Q_i - Q_i \mathscr{P}^{p^i}) y_3 = \mathscr{P}^{p^i} Q_i y_3 = \mathscr{P}^{p^i} e_i.$$

Here we note $|e_i| = 2(p^i - 1)$. Therefore

$$\begin{aligned} e_{i+1} &= \mathscr{P}^{\#-1} e_i = \mathscr{P}^{\#-1} f \mathscr{P}^{\#} y_8 + \mathscr{P}^{\#} f \mathscr{P}^{\#-1} y_8 \\ &= (\mathscr{P}^{\#-1} f) y_8^3 + f^3 y_8 y_{12}. \end{aligned}$$

Note $\mathscr{P}^{\#-1} X = 0$ and $\mathscr{P}^{\#-1} Y = 2y_{12}^3(-y_8^4) = y_{12}^3 y_8^4$ since $\mathscr{P}^5 y_{12} = -y_8^4$. Therefore

$$\mathscr{P}^{\#-1} \sum \lambda_{ij} X^i Y^j = \sum j\lambda_{ij} (X^i)^3 (Y^{j-1})^3 y_{12}^3 y_8^4.$$

This means $\mathscr{P}^{\#-1} f = (f')^3 Y X y_{12} y_8$ and $e_{i+1} = \{(f')^3 Y X^2 + f^3\} y_8 y_{12}$.

Next consider e_{i+2}. We have

$$\begin{aligned} e_{i+2} &= \mathscr{P}^{p^{i+1}} e_i = \mathscr{P}^{\#-1}(g y_8 y_{12}) \\ &= ((g')^3 Y X y_{12} y_8) y_{12}^3 y_8^3 + g^3 (y_8 y_{12}^4 + y_8^7) \\ &= y_8((g')^3 Y^3 X + g^3 (Y^2 + X^2)). \quad \square \end{aligned}$$

Lemma 4.11. *Let us write* $d_i = Q_i y_7$. *Then for* $i \geq 2$, $d_i = (e_{i-1})^3$. *In particular,* $d_i = y_8$, 0, y_8^3, $(y_{12}y_8)^3$ *for* $i = 0, 1, 2, 3$.

Proof. By induction, for $i \geq 2$,

$$Q_i d_i = \mathscr{P}^{p^i} d_{i-1} = \mathscr{P}^{p^i} (e_{i-2})^p = (\mathscr{P}^{\#-1} e_{i-2})^3. \quad \square$$

Lemma 4.12. *A greatest common divisor of* (e_i/y_8) *for all* $i \geq m$ *is equal to* 1.

Proof. We assume that f_i has no double root and X, Y as root. Then f_i and f_i' have no common divisor. Suppose g_{i+1} and f_i have same root. Then from Lemma 4.10, f_i and f'_i have same root and this contradicts to the first assumption. Since $g'_{i+1} = (f_i)^3X^2$, g_{i+1} and g'_{i+1} do not have the same divisor since so do not f_i and f'_i. Similar facts hold for $i+2$. □

Corollary 4.13. *The elements* $I'_1 = I_1/y_8$, $I'_2 = I_2/y_8^3$ *are prime for each* $m \geq 0$, *that is, for* $m \geq 0$ *if* $aI'_1 + bI'_2 = 0$ *in* $P(m)^* \otimes S_8 \otimes S_{12}$, *then* $a = a'I'_2$ *and* $b = -a'I'_1$ *in* $P(m)^* \otimes S_8 \otimes S_{12}$.

Proof. Note that $I'_1 = v_m f_m + v_{m+1} g_m y_{12} + \cdots$ and $I'_2 = v_m(g_{m-1}y_{12})^3 + v_{m+1}(f_m)^3 + \cdots$. If $I'_1 = ab$, then a is unit or b is unit from Lemma 4.12. These facts follow the corollary. □

Theorem 4.14. $P(m)^*(BPU(3)) \simeq P(m)^* \otimes_{BP^*} BP^*(BPU(3))$.

Proof. Suppose that $px = 0$ in $BP^*(BPU(3))$. Then $p\tilde{x} = aI_1 + bI_2$ in $BP^* \otimes S_8 \otimes S_{12}\tilde{y}_8^2$. Hence $0 = aI_1 + bI_2$ in $P(1)^* \otimes S_8 \otimes S_{12}\tilde{y}_8^2$. This means also $0 = aI'_1 + bI'_2$. Hence $a = I'_2a'$ and $b = -I'_1a' \mod p$. Therefore $a = I_2a' + pa''$, $b = -I_1a' + pb''$ in $BP^* \otimes S_8 \otimes S_{12}$. Hence $p\tilde{x} = pI_1a'' + pI_2b''$. This means $\tilde{x} = I_1a'' + I_2b''$ and $\tilde{x} = 0$ in $BP^*(BPU(3))$. Therefore there is no p-torsion in $BP^*(BPU(3))$. Hence when $m = 1$, the theorem is proved. The case $m \geq 2$ are also proved by the same argument from Corollary 4.13. □

Therefore $G = PU(3)$ satisfies (1)–(7) in the introduction.

5. $SO(4)$ for $p = 2$ and F_4 for $p = 3$

Recall $H^*(BSO(4); Z/2) = Z/2[w_2, w_3, w_4]$ and $Q_0w_2 = w_3$. It is known that there is no higher 2-torsion

(5.1) $$H^*(BSO(4))_{(2)} \simeq (Z_{(2)}[w_2^2] \oplus Z/2[w_3, w_2^2]\{w_3\}) \otimes Z_{(2)}[w_4].$$

Consider the Atiyah-Hirzebruch spectral sequence

(5.2) $$E_2^{*,*} = H^*(BSO(4); BP^*) \Rightarrow BP^*(BSO(4)).$$

From (2.6), $Q_1w_3 = w_3^2$ and $Q_1w_4 = w_4w_3$. Let us write $A = BP^*[w_2^2, w_4^2]$. Then from $d_{2p-1} = v_1 \otimes Q_1$, we have

(5.3) $$E_{2p}^{*,*} = A \oplus A/(2, v_1)[w_3^2]\{w_3^2, w_4w_3\}.$$

This module is a direct product of a free BP^*-module and a free $BP^*/(2, v_1)$-module. Hence the next nonzero differential is $d_{2p^2-1} = v_2 \otimes Q_2$. Since $Q_2w_4w_3 = w_3^2w_4^2$, we get

(5.4) $$E_{2p^2}^{*,*} = A\{1, 2w_4\} \oplus B\{w_3^2\}/(2, v_1) \oplus B[w_4^2]\{w_3^2w_4^2\}/(2, v_1, v_2)$$

where $B = BP^*[w_3^2, w_2^2]$. Since $E_{2p^2}^{*,*}$ is generated by even dimensional elements, we see $E_{2p^2} \simeq E_\infty$.

Theorem 5.5. *There is a* BP^**-module isomorphism*

$$BP^*(BSO(4)) \simeq \tilde{A}\{1, 2\tilde{w}_4\} \oplus \tilde{B}\{\tilde{w}_3^2\}/(I_1, I_2) \oplus \tilde{B}[[\tilde{w}_4^2]]\{\tilde{w}_3^2\tilde{w}_4^2\}/(I_1\tilde{w}_4^2, I_2\tilde{w}_4^2, I_3)$$

where $\tilde{A} = BP^*[[\tilde{w}_2^2, \tilde{w}_4^2]]$, $\tilde{B} = BP^*[[\tilde{w}_2^2, \tilde{w}_3^2]]$, *and* $I_1 \equiv \sum v_n\tilde{Q}_n(w_2w_3)$, $I_2 \equiv \sum v_n\tilde{Q}_n(w_3)$, *and* $I_3 \equiv \sum v_n\tilde{Q}_n(w_3w_4) \mod (2, v_1, \ldots)^2$.

Properties (1)–(3) and (7) hold immediately. Properties (4), (5) are proved by the arguments similar to the case $G = PU(3)$, but a little difficult.

Lemma 5.6. *Let* $I_1' = I_1/(\tilde{w}_3^2)$, $I_2' = I_2/(\tilde{w}_3^2)$, *and* $I_3' = I_3/(w_3^2 w_4^2)$. *The ideals* I_1' *and* I_2' *are prime in* $P(m)^* \otimes_{BP^*} B$. *If* $aI_3' \in \mathrm{Ideal}(I_2, I_1)$ *in* $P(m)^* \otimes_{BP^*} B[[w_4^2]]$, *then* $a \in \overline{\mathrm{Ideal}}(I_2, I_1)$.

Outline of proof. By the arguments similar to Lemma 4.10, we have $n \geq 2$

$$Q_n w_3 w_2 = (w_3 f(X, Y))^4 \quad \text{where } X = w_2^3, \ Y = w_3^2,$$
$$Q_{n+1} w_3 w_2 = (w_3 w_2 (f^2 + YX(f')^2))^4 \quad \text{where } f' = \partial f/\partial X,$$

and $Q_n w_3 = 2\sqrt{Q_{n+1}(w_3 w_2)}$. Hence we can prove I_1', I_2' are prime in $\widetilde{B} \otimes_{BP^*} P(m)^*$. Moreover we can see for $n \geq 2$

$$Q_n(w_3 w_4) = w_2^2 w_4^{2^{n-2}} \quad \mod w_2^4.$$

Suppose $aI_3' = b_1 I_1' + b_2 I_2'$ in $P(m)^* \otimes_{BP^*} \widetilde{B}[[\tilde{w}_4^2]]$. Here recall that (we assume m even)

$$I_1' = (v_m + v_{m+2}\tilde{w}_3^* + v_{m+4}\tilde{w}_3^* + \cdots)\tilde{w}_3^* \quad \mod \tilde{w}_2^2,$$
$$I_2' = (v_{m+1} + v_{m+3}\tilde{w}_3^* + \cdots)\tilde{w}_3^* \quad \mod \tilde{w}_2^2,$$
$$I_3' = (v_m + v_{m+1}\tilde{w}_4^* + v_{m+2}\tilde{w}_4^* + \cdots)\tilde{w}_4^* \quad \mod \tilde{w}_2^2.$$

Then we can easily see $a = b_1' I_1' + b_2' I_2' \mod \tilde{w}_2^2$. Now take out $(\tilde{a} = a - b_1' I_1' - b_2' I_2') I_3'$ from both sides of the supposition. Next, divide both sides by $\tilde{w}_2^2$. Using these arguments, we can prove this lemma. □

From Lemma 5.6, we can prove

(5.7) $$P(m)^*(BSO(4)) \simeq P(m)^* \otimes_{BP^*} BP^*(BSO(4))$$

and also prove (4), (5) in the introduction.

Remark 5.8. It is easily seen

$$BP^*(BSO(3)) \simeq BP^*(BSO(4))/(\tilde{w}_4^2, 2\tilde{w}_4)$$
$$\simeq \widetilde{\widetilde{A}} \oplus \widetilde{B}\{\tilde{w}_3^2\}/(I_1, I_2), \quad \text{where } \widetilde{\widetilde{A}} = BP^*[[\tilde{w}_2^2]].$$

Next consider $G = F_4$, $p = 3$. By Toda [T1] cohomology of $H^*(BF_4; Z/3)$ is known.

Theorem 5.9 (Toda).

$$H^*(BF_4; Z/3) \simeq Z/3[x_{36}, x_{48}] \otimes C$$

for

$$C = Z/3[x_4, x_8] \otimes \{1, x_{20}, x_{20}^2\} \oplus Z/3[x_{26}] \otimes \Lambda(x_9) \otimes \{1, x_{20}, x_{21}, x_{25}\}$$

where two terms of C *have the intersection* $\{1, x_{20}\}$.

Toda also determined the Steenrod algebra structure completely. (See Theorems I–III in [T1]). For example, $\beta x_i = x_{i+1}$ if x_{i+1} exists. Let $R_1 = Z/3[x_4, x_8]$, $R_2 = Z/3[x_8]$, and $R_3 = Z/3[x_{26}]$. Then it is easily computed in C

$$\mathrm{Ker}\, Q_0 = \{1\} \oplus x_4 R_1 \oplus x_8^2 R_2 \oplus x_{20} x_4 R_1 \oplus x_{20} x_8 R_2$$
$$\oplus x_{20}^2 R_1 \oplus x_{26} R_3 \oplus x_{21} R_3 \oplus x_9 R_3 \oplus x_9 x_{21} R_3$$

and

$$\operatorname{Ker} Q_0/\operatorname{Im} Q_0 \simeq \{1\} \oplus x_4 R_1 \oplus x_8^2 R_2 \oplus x_4 x_{20} R_1 \oplus x_8 x_{20} R_2 \oplus x_{20}^2 R_1.$$

The Poincaré series of $\operatorname{Ker} Q_0/\operatorname{Im} Q_0$ is

$$1 + \frac{t^4 + t^{24} + t^{40}}{(1-t^4)(1-t^8)} + \frac{t^{16} + t^{28}}{1 - t^8} = \frac{1}{(1-t^4)(1-t^{16})(1-t^{12})(1-t^{24})}.$$

Therefore we see

Proposition 5.10. *There is no higher torsion in* $H^*(BF_4)_{(3)}$.

Remark. This fact is also easily proved by using the Becker-Gottlieb transfer. For the fibering $\pi \to B\operatorname{Spin}(9) \xrightarrow{p} BF_4$, we have $p_* p^* = \times\chi(\pi) = \times 3$. Since $H^*(B\operatorname{Spin}(9))_{(3)}$ is 3-torsion free, there is no higher 3-torsion. This argument is also applied for $PU(3)$, $p = 3$ and E_8, $p = 5$.

Consider the Atiyah-Hirzebruch type spectral sequence $E_2^{*,*} = H^*(BF_4; BP^*) \Rightarrow BP^*(BF_4)$. Let $S_1 = BP^*[x_4, x_8]$, $S_2 = BP^*[x_8]$, $S_3 = BP^*/(3)[x_{26}]$, and $D = Z_{(3)}[x_{36}, x_{48}]$. Then

$$(5.11)\qquad E_2^{*,*} = (BP^*\{1\} \oplus S_1 x_4 \oplus S_2 x_8^2 \oplus S_1 x_4 x_{20} \oplus S_2 x_8 x_{20} \oplus S_1 x_{20}^2 \oplus S_3 \otimes \{x_{26}, x_{21}, x_9, x_9 x_{21}\}) \otimes D.$$

The first nonzero differential is $d_{2p-1} = v_1 \otimes Q_1$ and we know, from Toda, $Q_1 x_4 = x_9$, $Q_1 x_{20} = x_{25}$, $Q_1 x_{21} = x_{26}$. Let

$$A = (BP^*\{1, 3x_4\} \oplus S_1 x_4^2 \oplus S_2 x_8^2 \oplus S_1 x_4 x_{20} \oplus S_2 x_8 x_{20} \oplus S_1 x_{20}^2).$$

Then

$$(5.12)\qquad E_{2p}^{*,*} = (A \oplus S_3/(v_1)\{x_9, x_{26}\}) \otimes D.$$

Next nonzero differential is $d_{2p^2-1} = v_2 \otimes Q_2$ and $Q_2 x_9 = x_{26}$. Therefore

$$(5.13)\qquad E_{2p^2}^{*,*} = (A \oplus S_3/(v_1, v_2)\{x_{26}\}) \otimes D.$$

Since this is generated by even dimensional elements $E_{2p^2}^{*,*} \simeq E_\infty^{*,*}$. The properties (1)–(3), (7) hold from (5.13).

Theorem 5.14. *There is a* BP^**-module isomorphism*

$$BP^*(BF_4) \simeq \widetilde{A} \otimes \widetilde{D} \oplus BP^*[[\tilde{x}_{26}]]\{\tilde{x}_{26}\} \otimes \widetilde{D}/(I_1, I_2, I_3)$$

where $I_1 \equiv \sum v_n \widetilde{Q}_n(x_{25})$, $I_2 \equiv \sum v_n \widetilde{Q}_n(x_{21})$, *and*

$$I_3 \equiv \sum v_n \widetilde{Q}_n(x_9) \bmod (3, v_1, \ldots)^2.$$

The properties (4) and (5) are proved by the arguments similar to the arguments 4.11–4.14 and 5.6, but some more complicated. Note that $Q_n(x_i)$ are computed by Theorem III in [T]. For example, $Q_n(x_9) = 0,\ 0,\ x_{26}, x_{36}x_{26}, x_{26}(x_{36}^4 + x_{48}^3)$ for $n = 0, 1, 2, 3, 4$, respectively. For $n = \text{even} \geq 2$, $Q_n(x_9) = x_{26} f(X, Y)$ with $X = x_{36}^4$, $Y = x_{48}^3$. Then

$$Q_{n+1}(x_9) = x_{36}x_{26}((f')^3 X^2 Y + f^3) = x_{36}x_{26} g \mod x_{26}^2$$

and

$$Q_{n+2}(x_9) = x_{26}(YX^3(g')^3 + g^3(X + Y)) \mod x_{26}^2,$$

where $f' = \partial f/\partial X$.

6. $G = G_2, F_4, E_6$, AND $\mathrm{Spin}(n)$, $n \le 10$

The mod 2 cohomology of $B\,\mathrm{Spin}(n)$ is given by Quillen [Q2]

(6.1)
$$H^*(B\,\mathrm{Spin}(n); Z/2) \simeq Z/2[w_{2^h}(\Delta)] \otimes Z/2[w_2, \dots, w_n]/(Q_i w_2 | 0 \le i \le h)$$

where Δ is a spin representation of $\mathrm{Spin}(n)$ and 2^h is the Radon-Hurwitz number (see [Q2, p. 210]). When $n \le 9$, $H^*(B\,\mathrm{Spin}(n); Z/2)$ is a polynomial algebra generated by w_4, w_6, w_7, w_8, and $w_{2^h}(\Delta)$. Note that $G_2 \hookrightarrow \mathrm{Spin}(7)$ and

$$H^*(BG_2; Z/2) = Z/2[w_4, w_6, w_7]. \tag{6.2}$$

Cohomology BG_2 and BF_4 is given by Borel [B].

At first we study $BP^*(BG_2)$ and consider the Atiyah-Hirzebruch spectral sequence. Since $Q_0 w_6 = w_7$, we have

$$E_2^{*,*} = A \oplus A/(2)[w_7]\{w_7\} \quad \text{where } A = BP^*[w_4, w_6^2]. \tag{6.3}$$

Since $Q_1 w_4 = w_7$, we get for $B = BP^*[w_4^2, w_6^2]$

$$E_{2p}^{*,*} = B \oplus B\{2w_4\} \oplus B/(2, v_1)[w_7]\{w_7\}. \tag{6.4}$$

The facts $Q_2 w_7 = w_7^2$ and $d_{2p^2-1}(2w_4) \ne v_2 w_4 w_7$ because $d_{2p^2-1}(v_2 w_4 w_7) = v_2 w_7^2 \ne 0$, imply

$$E_{2p^2}^{*,*} = B \oplus B\{2w_4\} \oplus B/(2, v_1, v_2)[w_7^2]\{w_7^2\}. \tag{6.5}$$

Theorem 6.6. *$E_\infty^{*,*} \simeq E_{2p^2}^{*,*}$ and we get*

$$BP^*(BG_2) \simeq \widetilde{B} \oplus \widetilde{B}\{2\tilde{w}_4\} \oplus \widetilde{B}[[\tilde{w}_7^2]]\{\tilde{w}_7^2\}/(I_1, I_2, I_3)$$

where $I_1 = \sum v_n \widetilde{Q}_n(w_7 w_6)$, $I_2 = \sum v_n \widetilde{Q}_n(w_7 w_4)$, $I_3 = \sum v_n \widetilde{Q}_n(w_7)$, and $\widetilde{B} = BP^*[[\tilde{w}_4^2, \tilde{w}_6^2]]$. Hence $BP^*(BG_2)$ satisfies (1)–(3), (7).

Remark 6.7. The ideal (I_1, I_2, I_3) seems to satisfy the similar property in Lemma 5.6. However we cannot prove it yet.

Let us write by $E_r^{*,*}(BG)$ the E_r-term of the Atiyah-Hirzebruch type spectral sequence converging to $BP^*(BG)$.

Now we consider $B\,\mathrm{Spin}(n)$, $n = 7, 8, 9$, while $H^*(B\,\mathrm{Spin}(n); Z/2)$ for $n \le 6$ is generated by even dimensional elements. The cohomology

$$H^*(B\,\mathrm{Spin}(7); Z/2) \simeq H^*(BG_2; Z/2) \otimes Z/2[w_8] \tag{6.8}$$

and $Q_i w_8 = 0$ for $0 \le i \le 1$ and $Q_2 w_8 = w_8 w_7$. Therefore

$$E_r^{*,*}(B\,\mathrm{Spin}(7)) \simeq E_r^{*,*}(BG_2) \otimes Z_{(2)}[w_8] \quad \text{for } r \le 2p^2 - 2, \tag{6.9}$$

and we get

$$\begin{aligned} E_{2p^2}(B\,\mathrm{Spin}(7)) \simeq (&E_{2p^2}(BG_2) \oplus B/(2, v_1)\{w_8\} \oplus B\{2w_4 w_8\} \\ &\oplus B/(2, v_1, v_2)[w_7^2]\{w_8 w_7\})[w_8^2]. \end{aligned} \tag{6.10}$$

Since $Q_3 w_8 w_7 = w_8^2 w_7^2$, we have

$$\begin{aligned} E_{2p^3}(B\,\mathrm{Spin}(7)) \simeq (&E_{2p^2}(BG_2) \oplus B/(2, v_1)\{w_8\} \oplus B\{2w_2 w_8\} \\ &\oplus B/(2, v_1, v_2, v_3)[w_7^2]\{w_8^2 w_7^2\})[w_8^2]. \end{aligned} \tag{6.11}$$

Therefore $E_{2p^3} \simeq E_\infty$ and the properties (1)–(3), (7) hold for $G = \mathrm{Spin}(7)$.

The cohomologies are

(6.12) $$H^*(B\,\mathrm{Spin}(8); Z/2) \simeq H^*(B\,\mathrm{Spin}(7); Z/2) \otimes Z/2[w_8'],$$

and

$$H^*(B\,\mathrm{Spin}(9); Z/2) \simeq H^*(B\,\mathrm{Spin}(7); Z/2) \otimes Z/2[w_{16}].$$

We can compute

(6.13)

$$\begin{aligned} &E_{2p^4}(B\,\mathrm{Spin}(8)) \\ &\quad \simeq (E_{2p^3}(B\,\mathrm{Spin}(7)) \oplus E_{2p^3}(B\,\mathrm{Spin}(7))' \ominus E_{2p^3}(BG_2) \\ &\qquad \oplus (B(2, v_1, v_2)\{w_8w_8'\} \oplus B\{2w_4w_8w_8'\} \\ &\qquad\quad \oplus B/(2, v_1, v_2, v_3, v_4)[w_7^2]\{(w_8^4w_8'^2 + w_8^2w_8'^4)w_7^2\})[w_8^2])[w_8'^2], \end{aligned}$$

(6.14)

$$\begin{aligned} &E_{2p^4}(B\,\mathrm{Spin}(9)) \\ &\quad \simeq (E_{2p^3}(B\,\mathrm{Spin}(7)) \\ &\qquad \oplus (B(2, v_1, v_2)\{w_{16}\} \oplus B\{2w_4w_{16}\} \\ &\qquad\quad \oplus B(2, v_1)\{w_8w_{16}\} \oplus B\{2w_4w_8w_{16}\} \\ &\qquad\quad \oplus B/(2, v_1, v_2, v_3, v_4)[w_7^2]\{w_{16}^2w_8^2w_7^2\})[w_8^2])[w_{16}^2]. \end{aligned}$$

Here we note $Q_4Q_3(w_8w_8') = (w_8^4w_8'^2 + w_8^4w_8'^2)w_7^2$ and $Q_4Q_3w_{16} = w_8^2w_7^2w_{16}^2$. Hence $E_{2p^4} \simeq E_\infty$ and the properties (1)–(3) and (7) hold.

The cohomology is $H^*(BF_4; Z/2) \simeq H^*(BG_2; Z/2) \otimes Z/2[x_{16}, x_{24}]$. Moreover, $i^*\colon H^*(BF_4; Z/2) \to H^*(B\,\mathrm{Spin}(9); Z/2)$ is injective with $i^*x_{16} = w_{16} + \cdots$ and $i^*x_{24} = w_8^3 + \cdots$. We can see that $BP^*(BF_4)$ has the similar form as $BP^*(B\,\mathrm{Spin}(9))$ by exchanging w_8 for x_{24}. Therefore the properties (1)–(3), (7) hold for $G = F_4$.

The cohomology of $B\,\mathrm{Spin}(10)$ and E_6 are

(6.15) $$\begin{aligned} &H^*(B\,\mathrm{Spin}(10); Z/2) \\ &\quad \simeq H^*(B\,\mathrm{Spin}(7); Z/2) \otimes Z/2[w_{10}, w_{32}']/(w_7w_{10}), \end{aligned}$$

(6.16)

$$H^*(BE_6; Z/2) \simeq H^*(B\,\mathrm{Spin}(7); Z/2) \otimes Z/2[w_{10}, y_{18}, w_{32}', y_{34}, y_{48}']/R,$$

where R is the relation given Theorem 6.21 in [K-M2]. Since $Q_nw_{10} = 0$ and $Q_jw_{32}' = 0$ for $0 \le j \le 3$ from Theorem 6.7 in [Q2], we get

(6.17) $$\begin{aligned} &E_{2p^3}(B\,\mathrm{Spin}(10); Z/2) \\ &\quad \simeq (E_{2p^3}(B\,\mathrm{Spin}(7)) \oplus BP^*[w_6, w_8, w_{10}]\{w_{10}\}) \otimes Z_{(2)}[w_{32}']. \end{aligned}$$

By the similar reason, we get

(6.18)

$$\begin{aligned} E_{2p^3}(E_6; Z/2) &\simeq (E_{2p^3}(B\,\mathrm{Spin}(7)) \oplus BP^*[w_6, w_8, w_{10}, y_{18}]\{w_{10}, y_{18}, y_{34}\}) \\ &\quad \otimes Z_{(2)}[w_{32}', w_{48}']. \end{aligned}$$

Therefore $E_{2p^3} \simeq E_\infty$ and the properties (1)–(3) and (7) hold for $G = \mathrm{Spin}(10)$ and E_6.

At last we consider the case $G = PSU(4n+2)$. The cohomology is known from [K-M1]

(6.19) $$H^*(BPU(4n+2); Z/2) \simeq Z/2[a_2, a_3, x_{8k}, y(I)]/R$$

where $1 < k \le 2n$, $I = (i_1, \dots, i_r)$ for $1 < i_1 < \cdots < i_r \le 2n+1$, $|y_I| = 4\sum i_s - 2$, and R is the ideal generated by $a_3 y(I)$ and $y(I)^2 + \cdots$ and $y(I)y(J)$.

From Theorem 6.10 in [K-M1], x_{8k} is the $4k$th Chern class of representation to $U(\binom{4n+2}{2})$. Hence $Q_m x_{8k} = 0$ for all $m \ge 0$. Note that

$$0 = Q_m(a_3 y_I) = a_2^* a_3^* y_I + a_3 Q_m(y_I) = a_3 Q_m(y_I).$$

The $\ker \cdot a_3$ is generated by $(y(I))$, which is even dimensionally generated. Hence $Q_m(y_I) = 0$, for all $m \ge 0$. Therefore

(6.20) $\quad E_{2p}(BPU(4n+2)) \simeq E_{2p}(BSO(3)) \otimes Z_{(2)}[x_{8k}] \oplus BP^*[a_2, x_{8k}]\{y(I)\}.$

Hence (1)–(3), (7) hold also for these cases. A similar result holds for $G = PSp(2n+1)$ by using the result [K].

Adams conjectured that for all connected compact Lie group G, the map r in (3.2) is injective for $p \ge 3$

(6.21) $$r\colon H^*(BG; Z/p) \hookrightarrow \varprojlim_V H^*(BV; Z/p).$$

When $p = 2$ the above r is injective for $G = \mathrm{Spin}(8n+k)$, $k \equiv 1, 7, 8$ mod 8 by Quillen [Q] and for $G = SO(n)$, $O(n)$, G_2, F_4 by Borel [B] and for $G = E_6$ by Kono and Mimura [K-M2]. However for $G = E_7$ and $\mathrm{Spin}(11)$ the map r is not epic. The cohomology of $B\,\mathrm{Spin}(11)$ is given by

$$H^*(B\,\mathrm{Spin}(11); Z/2) \simeq Z/2[w_4, w_6, w_7, w_8, w_{10}, w_{11}]/R \otimes Z/2[w_{64}]$$

where $R = (w_{11}w_6 + w_{10}w_7,\ w_{11}^3 + w_{11}^2 w_4 w_7 + w_{11} w_8 w_7^2)$. Put

$$x = w_{10}^2 w_{11} + w_{10}^2 w_4 w_7 + w_{10} w_8 w_7 w_6 \in H^{31}(B\,\mathrm{Spin}(11); Z/2).$$

Then

$$w_{11}^2 x = w_{10}^2(w_{11}^3 + w_{11}^2 w_4 w_7 + w_{11} w_8 w_7^2) = 0.$$

Note that $x = w_{11}(w_{10}^2 + w_{10} w_6 w_4 + w_8 w_6^2)$ and hence $x^3 = 0$. On the other hand, define a map $\phi\colon H^*(B\,\mathrm{Spin}(11); Z/2) \to Z/2[a_{10}, a_{11}]/(a_{11}^3)$ by $\phi(w_j) = 0$ for $j = 4, 6, 7, 8, 64$ and by $\phi(w_{10}) = a_{10}$, $\phi(w_{11}) = a_{11}$. This map is a ring homomorphism and $\phi(x) = a_{10}^2 a_{11}$, $\phi(x)^2 \ne 0$. Therefore $x^2 \ne 0$ but $x^3 = 0$. Hence $r(x) = 0$.

Lemma 6.22. *If* (6.21) *holds, then Conjecture* 5 *holds, that is, for all odd dimensional elements* $x \in H^*(BG; Z/p)$, *there are* i *such that* $Q_m x \ne 0$ *for all* $m \ge i$.

Proof. The Q_m-homology $H(H^*(BV; Z/p); Q_m) \simeq \otimes Z/p[y_i]/(y_i^{p^m})$, $|y_i| = 2$ from Künneth formula. If $|x| \le |Q_m| = 2(p^m - 1)$, then $x \notin \operatorname{Im} Q_m$ and so $Q_m x \ne 0$.

Corollary 6.23. *If* (6.21) *holds, e.g.,* $G = SO(n)$, *then*

$$\rho(P(n)^*(BG) \to H^*(X; Z/p)) \subset H^{\mathrm{even}}(BG; Z/p),$$

for all $n \ge -1$ *(where* $P(-1) = BP$*).*

Remark 6.24. All examples given in §§5 and 6 satisfy the following conjecture stated in [T-Y]

$$BP\langle n-1\rangle^*(BG) \simeq BP\langle n-1\rangle^* \otimes_{BP^*} BP^*(BG) \quad \text{if } \mathrm{rank}_p G \le n.$$

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